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Journal Item

Received: 13 April 2020 | Revised: 22 November 2020 | Accepted: 15 February 2021

DOI: 10.1002/jcd.21773

RESEARCH ARTICLE

WILEY

Colouring problems for symmetric configurations with block size 3

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Funding information
Agentúra na Podporu Výskumu a Vývoja, Grant/Award Numbers: 15-0220, 17-0428; Vedecká Grantová Agentúra MŠVVaŠ SR a SAV, Grant/Award Numbers: 1/0142/17, 1/0238/19

Abstract

The study of symmetric configurations v_3 with block size 3 has a long and rich history. In this paper we consider two colouring problems which arise naturally in the study of these structures. The first of these is weak colouring, in which no block is monochromatic; the second is strong colouring, in which every block is multichromatic. The former has been studied before in relation to blocking sets. Results are proved on the possible sizes of blocking sets and we begin the investigation of strong colourings. We also show that the known 21_3 and 22_3 configurations without a blocking set are unique and make a complete enumeration of all nonisomorphic 20_3 configurations. We discuss the concept of connectivity in relation to symmetric configurations and complete the determination of the spectrum of 2-connected symmetric configurations without a blocking set. A number of open problems are presented.

KEYWORDS
blocking set, chromatic number, configuration

MATHEMATICAL SUBJECT CLASSIFICATION
05B30; 05C15

1 | INTRODUCTION

In this paper we will be concerned with symmetric configurations with block size 3 and, more particularly, two colouring problems which arise naturally from their study. First we recall the definitions. A *configuration* (v_r, b_k) is a finite incidence structure with v points and b blocks, with the property that there exist positive integers k and r such that

(i) each block contains exactly k points;
(ii) each point is contained in exactly r blocks; and
(iii) any pair of distinct points is contained in at most one block.

A configuration is said to be *decomposable* or *disconnected* if it is the union of two configurations on distinct point sets. We are primarily interested in indecomposable (connected) configurations, and so unless otherwise noted, this is assumed throughout the paper.

If $v = b$ (and hence necessarily $r = k$), the configuration is called *symmetric* and is usually denoted by v_k. We are interested in the case where $k = 3$. Such configurations include a number of well-known mathematical structures. The unique 7_3 configuration is the Fano plane, the unique 8_3 configuration is the affine plane $AG(2, 3)$ with any point and all the blocks containing it deleted, the Pappus configuration is one of three 9_3 configurations and the Desargues configuration is one of ten 10_3 configurations. Symmetric configurations have a long and rich history. It was Kantor in 1881 [17] who first enumerated the 9_3 and 10_3 configurations and in 1887, Martinetti [19] showed that there are exactly 31 configurations 11_3.

It is natural to associate two graphs with a symmetric configuration v_3. The first is the *Levi graph* or *point-block incidence graph*, obtained by considering the v points and v blocks of a configuration as vertices, and including an edge from a point to every block containing it. It follows that the Levi graph is a cubic (3-regular) bipartite graph of girth at least six. The second graph is the *associated graph*, obtained by considering only the points as vertices and joining two points by an edge if and only if they appear together in some block. Thus the associated graph is regular of valency 6 and order v.

We note that symmetric configurations with block size 3 have also been studied in the context of 3-regular, 3-uniform hypergraphs. In this scenario the points of the configuration are identified with the vertices in the hypergraph and the blocks with the hyperedges; the condition that no pair of distinct vertices should be in more than one hyperedge is usually referred to as a *linearity* condition in hypergraph terminology.

By a *colouring* of a symmetric configuration v_3, we mean a mapping from the set of points to a set of colours. In such a mapping, if no block is monochromatic we have a *weak colouring* and if every block is *multichromatic* or *rainbow* we have a *strong colouring*. The minimum number of colours required to obtain a weak (resp., strong) colouring will be called the *weak* (resp., *strong*) *chromatic number* and denoted by χ_w (resp., χ_s). It is immediate from the definition that the strong chromatic number χ_s of a configuration is equal to the chromatic number of its associated graph.

Weak colourings have been studied before in relation to so-called blocking sets and in Section 2.1 we begin the study of the sizes of these. In Section 2.2 we bring together various results concerning symmetric configurations without a blocking set which appears throughout the literature, some of which do not seem to be readily available. Section 2.3 is concerned with the connectivity of configurations and we complete the spectrum of 2-connected symmetric configurations without a blocking set. Our results on enumeration appear in Section 2.4.

In particular we extend known results by enumerating all symmetric configurations 20_3 together with their properties, and prove that the known 21_3 and 22_3 configurations without a blocking set are the unique configurations of those orders with that property. Section 3 is concerned with strong colourings. To the best of our knowledge, both this topic and the sizes of blocking sets in Section 2.1 appear to have been neglected and the results are new. Finally in Section 4 we bring together some of the open problems raised by the work in this paper.

2 | WEAK COLOURINGS

We begin with the following result which is a special case of Theorem 8 of [9].

Theorem 2.1 (Bollobás and Harris). *For every symmetric configuration v_3, either $\chi_w = 2$ or 3.*

A blocking set in a symmetric configuration is a subset of the set of points which has the property that every block contains both a point of the blocking set and a point of its complement. From this definition it is immediate that the complement of a blocking set is also a blocking set, and that the existence of a blocking set in a configuration is equivalent to $\chi_w = 2$. Empirical evidence indicates that almost all symmetric configurations v_3 contain a blocking set. Indeed, Table 2 shows that of the 122,239,000,083 connected configurations with $v \leq 20$, only 6 fail to have a blocking set.

For any $v \geq 8$, a configuration with a blocking set is very easy to construct. For v even, the set of blocks generated by the block $\{0, 1, 3\}$ under the mapping $i \mapsto i + 1 (\text{mod } v)$ has a blocking set consisting of all the odd numbers (and hence, another consisting of all the even numbers). For v odd and $v \geq 11$, construct the symmetric configuration $(v-1)_3$ as above and replace the blocks $\{0, 1, 3\}$ and $\{4, 5, 7\}$ with the blocks $\{1, 3, c\}$, $\{4, 7, c\}$ and $\{0, 5, c\}$, where c is a new point. The set of odd numbers is still a blocking set. The Fano plane does not have a blocking set, but all three 9_3 configurations do.

The above extension operation can be summarised and generalised as follows.

- Choose two nonintersecting blocks $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ such that the points a_1 and b_1 are not in a common block.
- Remove these blocks, introduce a new point c and add three new blocks $\{c, a_2, a_3\}$, $\{c, b_2, b_3\}$ and $\{c, a_1, b_1\}$.

This construction goes back to Martinetti [19]; see also [8].

2.1 | Sizes of blocking sets

Perhaps surprisingly, the cardinalities of blocking sets of those configurations v_3 for which $\chi_w = 2$ do not seem to have been studied. Let Q be such a blocking set and let $q = |Q|$. It is immediate that $\lceil v/3 \rceil \leq q \leq \lfloor 2v/3 \rfloor$. We have the following result.

Theorem 2.2. *Let v_3 be a symmetric configuration with a blocking set Q of cardinality q where $\lceil v/3 \rceil \leq q < \lfloor v/2 \rfloor$. Then v_3 also has a blocking set $\overline{Q}$ of cardinality $q + 1$.*

Proof. Let A, of cardinality α, be the set of blocks of the configuration which Q intersects in one point and B, of cardinality β, be the set of blocks which Q intersects in two points. Then $\alpha + \beta = v$ and $\alpha + 2\beta = 3q$. Thus since $q < \lfloor v/2 \rfloor$, $\beta = 3q - v < v - q$. Each block in B contains two points in Q and one point not in Q. Hence there exists a point $x \notin Q$ which is in no block of B, and so must be contained in three blocks of A. The set $\overline{Q} = Q \cup \{x\}$ is also a blocking set. □

Bearing in mind that if Q is a blocking set for a configuration v_3 then so is $V \backslash Q$, it follows that the range of cardinalities of blocking sets of a configuration is continuous, and that configurations can be categorised by the minimum cardinality of a blocking set; a blocking set of this minimum cardinality will be called a *minimal blocking set*. Configurations which have a blocking set of cardinality q for all q: $\lceil v/3 \rceil \leq q \leq \lfloor 2v/3 \rfloor$ are relatively easy to construct.

Theorem 2.3. *Let $v \geq 9$. Then there exists a configuration v_3 having a blocking set of cardinality q for all q: $\lceil v/3 \rceil \leq q \leq \lfloor 2v/3 \rfloor$.*

Proof. In view of Theorem 2.2, it is sufficient to construct a symmetric configuration with a blocking set of cardinality $\lceil v/3 \rceil$. Suppose first that $v \equiv 0 \pmod 3$, $v \geq 9$. Let $v = 3s$. Let the points of the configuration be $V = \{a_i, b_i, c_i\colon 0 \leq i \leq s-1\}$. Let the blocks be the sets $\{a_i, b_i, c_{i+1}\}$, $\{a_i, b_{i+1}, c_i\}$, $\{a_{i+1}, b_i, c_i\}$, $0 \leq i \leq s-1$, subscript arithmetic modulo s. The set $Q = \{a_i\colon 0 \leq i \leq s-1\}$ is a blocking set.

Now suppose that $v \equiv 1 \pmod 3$, $v \geq 10$. Construct a configuration $(v-1)_3$ as above. Introduce a new point ∞_0 and use Martinetti's extension operation, replacing the blocks $\{a_0, b_0, c_1\}$ and $\{a_1, b_1, c_2\}$ by blocks $\{\infty_0, b_0, b_1\}$, $\{\infty_0, a_0, c_1\}$ and $\{\infty_0, a_1, c_2\}$. The set $Q \cup \{b_1\}$ is a blocking set.

Finally, suppose that $v \equiv 2 \pmod 3$, $v \geq 11$. Construct a configuration $(v-1)_3$ as above. Introduce a further new point ∞_1 and again use the extension operation, replacing the blocks $\{a_0, b_1, c_0\}$ and $\{a_1, b_2, c_1\}$ by blocks $\{\infty_1, b_1, b_2\}$, $\{\infty_1, a_0, c_0\}$ and $\{\infty_1, a_1, c_1\}$. The set $Q \cup \{b_1\}$ is again a blocking set. □

We note that the condition $v \geq 9$ in the above theorem is necessary; the unique 7_3 configuration has no blocking set at all, and the unique 8_3 configuration has a minimal blocking set of cardinality 4. Since Theorem 2.3 shows that a configuration v_3 with a minimal blocking set as small as possible exists for all $v \geq 9$, it is natural to ask what the range of possible sizes of minimal blocking sets might be. At the minimum end of the range, we are able to prove the following results.

Theorem 2.4.

(a) *There exists a configuration v_3 with a minimal blocking set of size $\lceil \frac{v}{3} \rceil + 1$ for all $v \geq 8$.*
(b) *There exists a configuration v_3 with a minimal blocking set of size $\lceil \frac{v}{3} \rceil + 2$ for $v = 12$ and all $v \geq 15$.*

Proof. We deal first with part (a). First observe that from Table 1, there exists such a configuration v_3 for $8 \leq v \leq 16$. Let $\mathcal{A}$ be the set of all blocks of the configuration 8_3 as given in the appendix, that is, 012, 034, 056, 135, 147, 246, 257, 367. This has a minimal blocking set of size 4. Let $\mathcal{B}$ be the set of blocks of a configuration $(3s)_3$ as given in

TABLE 1 Sizes of minimum blocking sets of connected configurations

Points	Connected	Minimal blocking sets	
v	configurations	Size	Number
7	1	None	1
8	1	4	1
9	3	3	2
		4	1
10	10	4	8
		5	2
11	31	4	25
		5	6
12	229	4	45
		5	182
		6	2
13	2036	None	1
		5	2020
		6	15
14	21,398	5	16,884
		6	4514
		7	0
15	245,341	5	24,550
		6	220,720
		7	21
16	3,004,877	6	2,992,125
		7	12,750
		8	2
17	38,904,486	6	25,065,267
		7	13,839,209
		8	10

Theorem 2.3; that is, the points are the set $V = \{a_i, b_i, c_i\colon 0 \le i \le s-1\}$ and the blocks are the sets $\{a_i, b_i, c_{i+1}\}$, $\{a_i, b_{i+1}, c_i\}$, $\{a_{i+1}, b_i, c_i\}$, $0 \le i \le s-1$, subscript arithmetic modulo s. Replace the block $\{0, 1, 2\}$ by $\{a_0, 1, 2\}$ and the block $\{a_0, b_0, c_1\}$ by $\{0, b_0, c_1\}$ to form sets $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, respectively. The set $\overline{\mathcal{A}} \cup \overline{\mathcal{B}}$ is a connected configuration $(3s+8)_3$. We need to show that this has a minimal blocking set of size 4.

Considering the set $\overline{\mathcal{A}}$, a blocking set Q must contain at least four points of the set $\{a_0, i\colon 0 \le i \le 7\}$ and further, in the special case that both $a_0, 0 \in Q$ it must contain at least five points. Otherwise, then by replacing the point a_0 by the point 0 to return to the set $\mathcal{A}$, the configuration 8_3 would have a blocking set of size 3. Now consider the set $\overline{\mathcal{B}}$. In the above special case, Q must contain at least $s-1$ elements of the set $V\setminus\{a_0\}$ and in all other cases, at least s elements. So Q has at least $s+4$ elements; to show that a minimal blocking set has exactly $s+4$ elements we may take a blocking set $Q=\{1,4,5,6,b_i\colon 0\le i\le s-1\}$.

We next deal with configurations $(3s+9)_3$, $s\ge 3$. The procedure is precisely the same as the above case, except that we use the configuration 9_3 as given in the appendix, that is, 012, 034, 056, 135, 147, 248, 267, 368, 578 which also has a minimal blocking set of size 4. In this case we take a blocking set $Q=\{1,4,5,6,b_i\colon 0\le i\le s-1\}$.

Finally for configurations $(3s+10)_3$ we use one of the two configurations 10_3 as given in the appendix with a minimal blocking set of size 5, namely, 012, 034, 056, 135, 178, 247, 268, 379, 469, 589. Again the procedure is as in the above two cases and we can take a blocking set $Q=\{1,4,5,6,7,b_i\colon 0\le i\le s-1\}$.

Now we deal with part (b). From Table 1, there exists such a configuration for $v\in\{12,15,16,17\}$. The configuration on the set $\mathbb{Z}_v$ generated by the block $\{0,1,3\}$ under the mapping $i\mapsto i+1 \pmod v$ has a blocking set of size $\lceil\frac{v}{3}\rceil+2$ for $v\in\{18,21,22,23,24\}$; see Theorem 2.5. The case where $v=19$ is of particular interest since $\lceil\frac{v}{3}\rceil+2=\lfloor\frac{v}{2}\rfloor$, the maximum size of a minimal blocking set. Of the 7,597,039,898 connected configurations 19_3, see [12] and Table 2, only seven have a minimal blocking set of size 9 and these are given below.

TABLE 2 Numbers of configurations v_3

v	a	b	c	d	e	f	g	h	i
7	1	1	1	1	1	1	1	1	0
8	1	1	1	1	1	1	1	0	0
9	3	3	3	2	1	1	1	0	0
10	10	10	10	2	1	1	1	0	0
11	31	25	25	1	1	0	0	0	0
12	229	95	95	4	3	1	1	0	0
13	2036	366	365	2	2	1	1	1	0
14	21,399	1433	1432	3	3	1	1	0	1
15	245,342	5802	5799	5	4	1	1	0	1
16	3,004,881	24,105	24,092	6	4	2	2	0	4
17	38,904,499	102,479	102,413	2	2	0	0	0	13
18	530,452,205	445,577	445,363	9	5	1	1	0	47
19	7,597,040,188	1,979,772	1,979,048	3	3	1	1	4	290
20	114,069,332,027	8,981,097	8,978,373	9	5	2	2	0	2413

Notes: a is the number of configurations v_3; b is the number of self-dual configurations; c is the number of self-polar configurations; d is the number of point-transitive configurations; e is the number of cyclic configurations; f is the number of flag-transitive configurations; g is the number of weakly flag-transitive configurations; h is the number of connected blocking set-free configurations; i is the number of disconnected configurations.

```
012 034 056 137 189 25a 28b 3cd 46e 4cf 5gh 6gi 79h 7dg
  8ei 9ef abc afh bdi
012 034 056 137 145 236 258 469 7ab 7cd 8ae 8cf 9ag 9ch
  bdi beg dfh efi ghi
012 034 056 137 158 239 2ab 46c 47d 5ef 6eg 78h 8fi 9af
  9ei acd bcg bdh ghi
012 034 056 137 158 239 2ab 47c 4de 5fg 68f 6hi 7dh 8ei
  9ag 9bi acf beh cdg
012 034 056 137 148 239 24a 5bc 5de 6bd 6cf 78g 79h 8ah
  9ag bef cei dfi ghi
012 034 056 137 158 239 2ab 457 46c 6de 78f 8gh 9ai 9bg
  acd bdh cei efh fgi
012 034 056 137 145 236 257 468 79a 89b 8ac 9de adf bcg
  beh cfi dhi egi fgh
```

The most interesting of these is possibly the first one which is point-transitive; one of only three such configurations 19_3, again see [12] and Table 2. Its Levi graph is arc-regular and has automorphism group of order 114. It is the unique symmetric graph of order 38 and is graph F038A in the Foster census [13]. The configuration is cyclic and is isomorphic to the configuration generated by the block $\{0, 1, 8\}$ under the mapping $i \mapsto i + 1 \pmod{19}$. An example of a symmetric configuration on 20 points having a minimal blocking set of size 9 is as follows.

```
012 034 056 135 146 237 245 678 79a 8bc 8de 9bf 9dg abh adi
  cej cfh egi fgj hij
```

So we may assume that $v \geq 25$. We follow closely the argument above. Let $\mathcal{A}$ and $\mathcal{A}'$ be the sets of all blocks of the configuration 8_3 as given in the appendix on point sets $\{0, 1, ..., 7\}$ and $\{0', 1', ..., 7'\}$, respectively. Let $\mathcal{B}$ be as in part (a). Replace the block $\{0, 1, 2\}$ by $\{a_0, 1, 2\}$, the block $\{0', 1', 2'\}$ by $\{b_0, 1', 2'\}$ and the blocks $\{a_0, b_0, c_1\}$ and $\{a_1, b_0, c_0\}$ by $\{0, b_0, c_1\}$ and $\{a_1, 0', c_0\}$ to form sets $\overline{\mathcal{A}}$, $\overline{\mathcal{A}'}$ and $\overline{\mathcal{B}}$, respectively.

As in part (a), by considering the sets $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}'}$, a blocking set Q must contain at least four points of each of the sets $\{a_0, i: 0 \leq i \leq 7\}$ and $\{b_0, i': 0 \leq i \leq 7\}$; and if both $a_0, 0 \in Q$ at least five points of the former set and if both $b_0, 0' \in Q$ at least five points of the latter set. Now by considering the set $\overline{\mathcal{B}}$, Q must contain at least $s - 2$ elements of the set $V \backslash \{a_0, b_0\}$ if $\{a_0, 0, b_0, 0'\} \subset Q$; $s - 1$ elements if either $a_0, 0 \in Q$ or $b_0, 0' \in Q$ but not both; and s elements otherwise. In other words, Q must contain at least $s + 8$ elements in total. To show that a minimal blocking set has exactly $s + 8$ elements, take $Q = \{1, 4, 5, 6, 1', 4', 5', 6', c_i: 0 \leq i \leq s - 1\}$. This deals with symmetric configurations $(3s + 16)_3$, $s \geq 3$.

To deal with symmetric configurations $(3s + 17)_3$, $s \geq 3$, the procedure is precisely the same except that for the set $\mathcal{A}$ we use the configuration 9_3 as given in the appendix. A minimal blocking set of size $s + 8$ is again $Q = \{1, 4, 5, 6, 1', 4', 5', 6', c_i: 0 \leq i \leq s - 1\}$. Finally for configurations $(3s + 18)_3$, $s \geq 3$, we also replace the set $\mathcal{A}'$ with the configuration 9_3 on point set $\{0', 1', ..., 8'\}$. Again a minimal blocking set of size $s + 8$ is $Q = \{1, 4, 5, 6, 1', 4', 5', 6', c_i: 0 \le i \le s - 1\}$. □

At the maximum end of the range, the situation appears to be much more difficult. Table 1 shows minimal blocking set sizes for symmetric configurations with $v \le 17$. Of the 42,178,413 such configurations, only 60 have a minimal blocking set of size $\lfloor \frac{v}{2} \rfloor$; the 27 examples for $v \le 14$ are shown in the appendix. At $v = 16$, it is noteworthy that only two of the very large number of configurations fail to have a blocking set of size 7; these are shown below.

```
012 034 156 078 59a 9bc 3de 57f 4bd 26b ace 8ef 479 13a 28c 6df
012 034 567 589 0ab cde 6cf 136 78d 2ad 9ef 49b 37c 5bf 28e 14a
```

The first of these is one of the two flag-transitive configurations on 16 points; see [6] and Table 2. Indeed its Levi graph is the Dyck graph, which is well known and is the unique arc-transitive cubic graph on 32 vertices. The Dyck graph is graph F032A in the Foster census [13]. Although this graph has a number of known constructions, it seems that none of these can be generalised to produce further examples of configurations without small blocking sets. Unfortunately therefore, we cannot provide a construction for an infinite class of symmetric configurations v_3 having a minimal blocking set of maximum cardinality, and this remains a significant open problem.

However, we are able to construct symmetric configurations whose minimal blocking sets have a size as far away as we please from both the minimum or maximum possible cardinalities, as the following theorem and corollary show.

Theorem 2.5. *Let $v \ge 8$ and let C_v be the cyclic configuration on v points generated by the block* $\{0, 1, 3\}$ *under the mapping $i \mapsto i + 1$* $(\text{mod } v)$. *Then the size of a minimal blocking set in C_v is*

$$m(v) = 2\left\lfloor \frac{v}{5} \right\rfloor + \varepsilon, \quad \text{where } \varepsilon = \begin{cases} 0 & \text{if } v \equiv 0 \pmod 5, \\ 1 & \text{if } v \equiv 1 \pmod 5, \\ 2 & \text{if } v \equiv 2, 3, 4 \pmod 5). \end{cases}$$

The proof of Theorem 2.5 is much simplified by transforming the problem into an equivalent problem concerning the existence of binary words. A *binary word* $\mathbf{b}$ of length n is a sequence $b_0, b_1, ..., b_{n-1}$ where each $b_i \in \{0, 1\}$. We shall be concerned with *circular* binary words, where the digit b_0 is considered to follow b_{n-1}; informally, the word "wraps round" with period n. A *subword* of length m is a sequence $b_i, b_{i+1}, ..., b_{i+m-1}$ where the subscripts are taken mod n; in other words, the subword starts at position i and wraps round if necessary. The *weight* $w(\mathbf{b})$ of a word $\mathbf{b}$ is simply the number of 1s in $\mathbf{b}$. A sequence of k consecutive 1s in a circular binary word with 0s at either end is called a *run of length k*; similarly for a sequence of 0s surrounded by 1s.

To make the connection with blocking sets, we let C_v be a cyclic configuration as in the statement of the theorem, and identify the point set V of C_v with the elements $\{0, 1, ..., v - 1\}$ of the cyclic group $\mathbb{Z}_v$. To each subset $S \subseteq V$ we identify a binary word $\mathbf{b}(S) = b_0, b_1, ..., b_{v-1}$ where $b_i = 1$ if $i \in S$ and $b_i = 0$ otherwise. Since each block of C_v has the form $\{m, m + 1, m + 3\}$, it is immediate that a subset S is a blocking set for C_v if and only if the corresponding circular binary word $\mathbf{b}(S)$ does not contain any of the subwords 0000, 0010, 1101 or 1111. The problem of finding a minimal blocking set is therefore equivalent to finding the minimum weight of a circular binary word satisfying this forbidden subword criterion. We begin with two simple lemmas.

Lemma 2.6. *Suppose* $\mathbf{b} = \mathbf{b}(S)$ *is a circular binary word corresponding to a blocking set* S *of the configuration* C_v*. Then any subword of* $\mathbf{b}$ *of length* 5 *has weight* 2 *or* 3.

Proof. Clearly any subword of length 5 and weight 0 contains the forbidden subword 0000. It is easy to see that the only possible length 5 subword of weight 1 is 01000. The digit immediately to the left of this subword must be 1, otherwise we get the forbidden subword 0010. Then the next digit to the left again must be 0, to avoid the forbidden subword 1101. Continuing in this way, we see that the sequence of digits reading leftwards from 01000 must be $1, 0, 1, 0, 1, 0, \ldots$. But $\mathbf{b}$ is a circular word containing the subword 000, so this is impossible. Thus no subword of length 5 can have weight 1.

Since the roles of the binary digits 0 and 1 in this problem are symmetric (corresponding to the fact that if S is a blocking set then so is its complement), it follows that no length 5 subword can have weight 4 or 5 either. □

Lemma 2.7. *Suppose* $\mathbf{b} = \mathbf{b}(S)$ *is a circular binary word corresponding to a blocking set* S *of the configuration* C_v*. Then* $\mathbf{b}$ *has one of the following forms*:

(a) 01010101... *or* 10101010... (*possible only if* v *is even*);
(b) *a sequence of* 0*s and* 1*s in runs of length* 2 *or* 3 *only.*

Proof. The proof of Lemma 2.6 shows that whenever the subword 010 appears in $\mathbf{b}$, then $\mathbf{b}$ must be of type (a). A similar argument holds for the subword 101. Thus any run length of 1 forces type (a), and this is only possible if v is even. Run lengths of 4 or greater are ruled out by the forbidden subwords 0000 and 1111, so the only remaining possibility is type (b). □

We are now ready to complete the proof of the theorem.

Proof of Theorem 2.5. Suppose $\mathbf{b} = \mathbf{b}(S)$ is a circular binary word corresponding to a blocking set S of the configuration C_v. A simple counting argument in conjunction with Lemma 2.6 shows that $|S| = w(\mathbf{b}) \geq \frac{2v}{5}$. So writing $|S| = 2\lfloor \frac{v}{5} \rfloor + \varepsilon$, it remains to find the minimum value of ε in all cases. We proceed by considering all the congruence classes mod 5.

If $v \equiv 0 \pmod 5$, then $\mathbf{b} = 11000\ 11000\ 11000\ldots$ satisfies the conditions of Lemma 2.7 and so $\varepsilon = 0$.

If $v \equiv 1 \pmod 5$, then we know $\varepsilon \geq 1$ and $\mathbf{b} = 11000\ 11000\ldots 11000\ 1$ satisfies the conditions of Lemma 2.7 and so $\varepsilon = 1$.

If $v \equiv 2 \pmod 5$, then we know $\varepsilon \geq 1$ but an examination of all the possibilities shows that it is not possible to add a single 1 and a single 0 to a word of the form $\mathbf{b} = 11000\ 11000\ 11000\ldots$ without creating a run of length 1 or 4. Thus $\varepsilon \geq 2$, and $\mathbf{b} = 111000\ 111000\ 11000\ldots 11000$ satisfies the conditions of Lemma 2.7 and so $\varepsilon = 2$.

If $v \equiv 3 \pmod 5$, then we know $\varepsilon \geq 2$ and $\mathbf{b} = 1100\ 1100\ 11000 \ldots 11000$ satisfies the conditions of Lemma 2.7 and so $\varepsilon = 2$.

If $v \equiv 4 \pmod 5$, then we know $\varepsilon \geq 2$ and $\mathbf{b} = 11000\ 1100\ 11000 \ldots 11000$ satisfies the conditions of Lemma 2.7 and so $\varepsilon = 2$. □

Corollary 2.8. *Let $k \geq 1$. Then there exist*:

(a) *a configuration v_3 with a minimal blocking set of size exactly* $\left\lceil \frac{v}{3} \right\rceil + k$; and

(b) *a configuration v_3 with a minimal blocking set of size exactly* $\left\lfloor \frac{v}{2} \right\rfloor - k$.

Proof. We use Theorem 2.5. For (a), take $v = 15k$ and for (b), take $v = 10k$. □

2.2 | Configurations without blocking sets

We now turn our attention to the case where $\chi_w = 3$, that is, to symmetric configurations with block size 3 which have no blocking set. This is an old problem which goes back some 30 years. It has appeared three times as a problem at the British Combinatorial Conference. The first time was as Problem 194 in the Proceedings of the 13th Conference [3], proposed by H. Gropp and originated by J. W. DiPaola and H. Gropp. At that time there were 13 unresolved values: 15, 16, 17, 18, 20, 23, 24, 26, 29, 30, 32, 38, 44. It appeared again as Problem 228 in the next Proceedings [4], by which time the five largest values had been resolved positively due to the work of Kornerup [18]. Finally in the Proceedings of the 16th Conference [5], Problem 333, Gropp asked whether there exists a symmetric configuration 16_3 without a blocking set, having reported that the case of such a configuration 15_3 had been resolved negatively.

All configurations v_3 for $7 \leq v \leq 18$ were enumerated by Betten, Brinkmann and Pisanski [6] in a paper published in 2000, leaving only the values 20, 23, 24, 26 unresolved. The problem was finally solved in 2003 by Funk et al. [14]. A bipartite graph G with bipartition $\{X, Y\}$ such that $|X| = |Y| = n$ is said to be *det-extremal* if its $n \times n$ biadjacency matrix A satisfies the equation $|\det(A)| = \mathrm{per}(A)$. (In our context, the biadjacency matrix of the Levi graph of a configuration v_3 is simply the $v \times v$ incidence matrix of the configuration.) Thomassen [23] pointed out that a symmetric k-configuration is blocking set free if and only if its Levi graph is det-extremal. In [14] the following theorem was proved from which it is an immediate corollary that there are no symmetric configurations v_3 without a blocking set for $v = 20, 23, 24, 26$.

Theorem 2.9 (Funk, Jackson, Labbate and Sheehan). *There exists a det-extremal connected cubic bipartite graph of order $2v$ if and only if $v \in \{7, 13, 19, 21, 22, 25\}$ or $v \geq 27$.*

The four values 20, 23, 24, 26 indeed seem to be the most problematic. If $v \geq 27$, it is easy to give a short self-contained account to prove that there exists a symmetric configuration v_3 with no blocking set and we do this below beginning with two constructions from [9] which we present as theorems.

Theorem 2.10 (Bollobás and Harris). *If there exist configurations v_3 and $(v')_3$ without a blocking set, then there exists a configuration $(v + v' - 1)_3$ without a blocking set.*

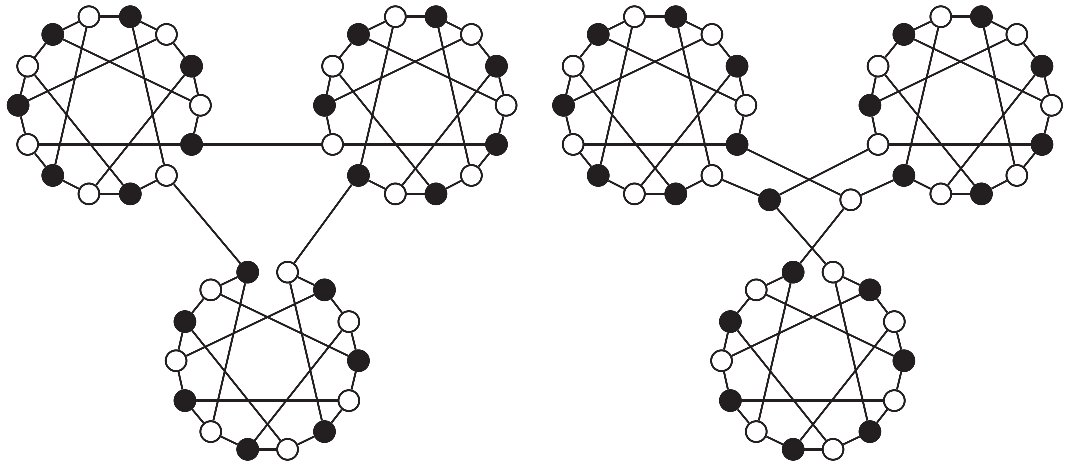

FIGURE 1 Levi graphs of the unique configurations 21_3 and 22_3 with no blocking set

Proof. Denote the points and blocks of the configuration v_3 (resp., $(v')_3$) by V and $\mathcal{B}$ (resp., V' and $\mathcal{B}'$). Choose $B \in \mathcal{B}$ and suppose that points $x_1, x_2, x_3 \in B$. Further choose $x' \in V'$ and suppose that x' is contained in blocks B'_1, B'_2, B'_3. Define new blocks $B_i'' = (B'_i \backslash \{x'\}) \cup \{x_i\}$, $i = 1, 2, 3$. Then $V \cup (V' \backslash \{x'\})$ and $(\mathcal{B} \backslash \{B\}) \cup (\mathcal{B}' \backslash \{B'_1, B'_2, B'_3\}) \cup \{B_1'', B_2'', B_3''\}$ are the points and blocks of a configuration $(v + v' - 1)_3$ which it is easily verified has no blocking set. □

Theorem 2.11 (Bollobás and Harris). *If there exist configurations* $(v^1)_3, (v^2)_3, \ldots, (v^{2k+1})_3$ *without a blocking set, then there exists a configuration* $(v^1 + v^2 + \cdots + v^{2k+1})_3$ *without a blocking set.*

Proof. Denote the points and blocks of the configuration $(v^i)_3$ by V^i and $\mathcal{B}^i$, respectively, $i = 1, 2, \ldots, 2k + 1$. For each i choose a block $B^i \in \mathcal{B}^i$ and a point $x^i \in B^i$. Define a new block $B_*^i = (B^i \backslash \{x^i\}) \cup \{x^{i+1}\}$, superscript arithmetic modulo $2k + 1$. Then $\bigcup_{i=1}^{2k+1} V^i$ and $\bigcup_{i=1}^{2k+1} (\mathcal{B}^i \backslash \{B^i\}) \cup \{B_*^i\}$ are the points and blocks of a configuration $(v^1 + v^2 + \cdots + v^{2k+1})_3$. Again it is easy to verify that this has no blocking set. □

We note that the construction of Theorem 2.11 was reported independently by Abbott and Hare [1], referencing an earlier paper of Abbott and Liu [2].

To implement the constructions we begin with three basic systems. From [6], in the range $7 \le v \le 18$ there exist only two symmetric configurations v_3 with no blocking set: the unique 7_3 configuration (Fano plane) and a 13_3 configuration obtained from two copies of it using Theorem 2.10.

The blocks of the latter system can be represented by the following triples:

```
012 034 056 135 146 236 278 49c 5ab 79b 7ac 89a 8bc.
```

A symmetric configuration 22_3 with no blocking set was given by Dorwart and Grünbaum [11]; it is illustrated in Figure 1 and as is evident, is obtained by merging three Fano planes. Its blocks are as follows:

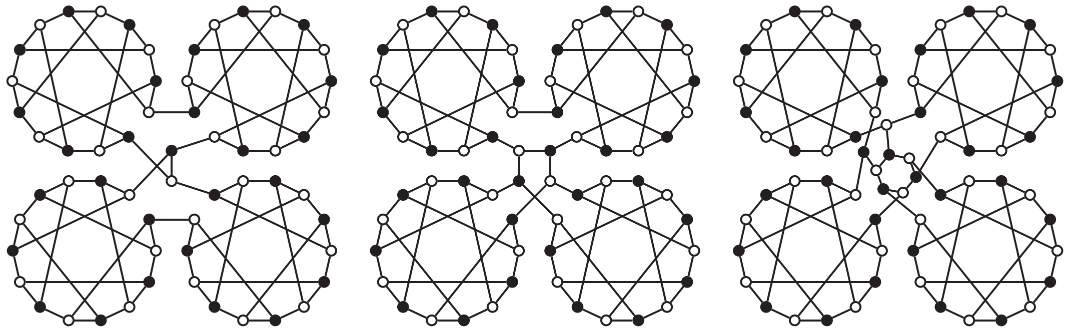

FIGURE 2 Levi graphs of configurations 29_3, 30_3 and 32_3 with no blocking set

```
012 034 056 135 146 236 24l 58f 79c 7ak 7bl 89a 8ck 9bk abc deh
  dfj dgi egj eil fgh hij
```

To deduce the existence of blocking set free configurations v_3 for all $v \geq 27$, first note that by putting $v' = 7$ in Theorem 2.10, it follows that if there exists a blocking set free configuration v_3 then there exists a blocking set free configuration $(v + 6)_3$. Thus once there exist such configurations for six consecutive values of v, existence for all larger values of v follows inductively. Existence for the value $v = 27$ follows from Theorem 2.11 by putting $v^1 = v^2 = 7$ and $v^3 = 13$, and for the value $v = 28$ from Theorem 2.10 by putting $v = 7$ and $v' = 22$. For the value $v = 31$, first construct a configuration 25_3 from Theorem 2.10 by putting $v = v' = 13$ and then, again from Theorem 2.10, by putting $v = 7$ and $v' = 25$.

As reported in [16], Kornerup [18] constructed blocking set free configurations v_3 for the values $v = 29, 30, 32$. These are contained in a thesis of the University of Aarhus which we have been unable to see, and the configurations found do not seem to be published elsewhere. Thus to give a complete account in one place, we have also constructed configurations 29_3, 30_3 and 32_3 without a blocking set. We show their Levi graphs in Figure 2, and include the blocks below.

A blocking set free configuration 29_3:

```
012 034 056 135 146 29d 2bc 367 457 7es 89b 8as 8cd 9ac abd egk
eij fgifhl fjk ghj hik lnr lpq mnp mos mqr noq opr
```

A blocking set free configuration 30_3:

```
012 034 056 135 146 2bt 2fm 367 457 789 8ae 8cd 9ac 9de abd bce
fhl fjkghj git gkl hik ijl mos mqr noq npt nrs opr pqs
```

A blocking set free configuration 32_3:

```
012 034 056 135 146 27k 2tv 36l 45l 78a 7bc 89b 8ck 9ac 9gt abk
deu dfjdhi efh eij fgi ghj luv mnv mos mqr noq nrs opr pqs ptu
```

Again we have used the “merging” technique and claim no originality for these. They may very well be the same systems discovered by Kornerup.

2.3 | Connectivity of configurations

Here we introduce the idea of the connectivity of a symmetric configuration and derive some results. First recall that in a cubic graph, the vertex connectivity is equal to the edge connectivity. Further if a connected cubic graph is also bipartite, then the connectivity cannot be 1 and so is equal to either 2 or 3. Define the *connectivity* of a symmetric configuration to be the connectivity of its Levi graph. Funk et al. [14] present the following operation. Let G_1 and G_2 be cubic bipartite graphs which are disjoint, and let $y \in V(G_1)$ with neighbour set $\{x_1, x_2, x_3\}$ and $x \in V(G_2)$ with neighbour set $\{y_1, y_2, y_3\}$. Then the graph

$$G = (G_1 \backslash y) \cup (G_2 \backslash x) \cup \{x_1y_1, x_2y_2, x_3y_3\}$$

is said to be a *vertex-sum* of G_1 and G_2. They then quote the following theorem which they attribute to McCuaig [20].

Theorem 2.12 (McCuaig)**.** *A 3-connected cubic bipartite graph is det-extremal if and only if it can be obtained from the Heawood graph by repeatedly applying the vertex-sum operation.*

For our purposes, the significance of the vertex-sum operation on cubic bipartite graphs is that it is equivalent to the $v + v' - 1$ construction of Bollobás and Harris given in Theorem 2.10. Thus we have the following result.

Theorem 2.13. *A* 3*-connected symmetric configuration* v_3 *without a blocking set exists if and only if* $v \equiv 1 \pmod 6$*. Moreover, such systems can only be obtained from the Fano plane by repeatedly applying the* $v + v' - 1$ *construction.*

This naturally raises the question of the spectrum of 2-connected symmetric configurations without a blocking set. From our account above it is clear that the systems v_3 with $v \equiv 1 \pmod 6$ arising from Theorem 2.13 are 3-connected. There are no 2-connected systems for $v \in \{7, 13, 19\}$ since all have been enumerated and arise from Theorem 2.13; see Table 2 and

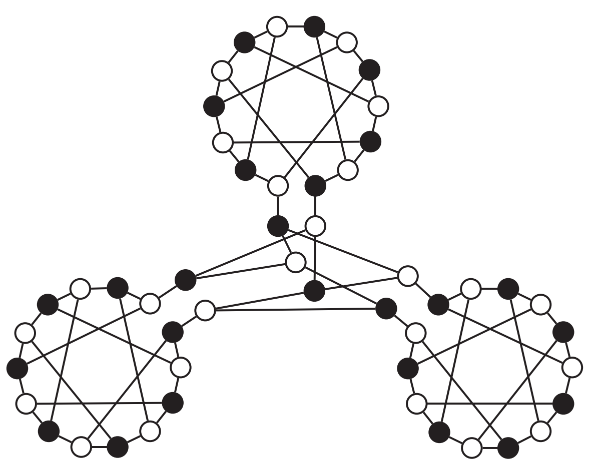

FIGURE 3 The Levi graph of a 25_3 configuration with no blocking set

the discussion in Section 2.4. So to complete the spectrum, what is needed is a 2-connected configuration 25_3 without a blocking set. Such a configuration does exist and its Levi graph is shown in Figure 3. The blocks are listed below.

```
012 034 056 135 146 236 24m 5ln 78d 79c 7an 89b 8ac 9ad bcd blo ego ehi
ejk fhk fij flm ghj gik mno
```

In fact this graph has already appeared in the literature. It appears as fig. 8 in [20] as a 2-connected unbalanced 1-extendible cubic bipartite graph. We have the following result.

Theorem 2.14. *A 2-connected symmetric configuration v_3 without a blocking set exists if and only if $v \in \{21, 22, 25\}$ or $v \geq 27$.*

2.4 | Enumeration of configurations

Finally in this section we present some enumeration results. As stated above, for $7 \leq v \leq 18$ there exist just two symmetric configurations with no blocking set; unique 7_3 and 13_3 systems. Gropp [16] reported that there exist at least four configurations 19_3 without a blocking set. Recently the present authors [12] have enumerated all configurations 19_3 and we confirm that there are exactly four without a blocking set. These have a nice description as follows. Because $19 = 13 + 7 - 1$, it must be true that at least some of the four configurations 19_3 can be obtained by using Theorem 2.10 with the unique 13_3 and 7_3 configurations without blocking sets. We may use the construction of Theorem 2.10 with $v = 7, v' = 13$, taking all possible choices for the distinguished point and block in the two constituent configurations. To this set of configurations we may add those obtained by taking $v = 13, v' = 7$ in the same way. Finally, this set of configurations can be reduced to isomorphism class representatives using the GAP package DESIGN [15,22]. In this way we were able to determine that the method results in exactly four isomorphism classes of configurations 19_3 with no blocking set. Thus these correspond precisely to the four in the enumeration; this description of the four 19_3 configurations was known to Gropp and the construction is described in [10,16]. The blocks of these four 19_3 systems are as follows:

```
012 034 056 135 19a 236 245 4bc 678 79b 7ac 89c 8de afi bgh dfh
  dgi efg ehi
012 034 056 135 146 29a 2bc 367 4gh 5fi 79b 7ac 89c 8ab 8de dfh
  dgi efg ehi
012 034 056 135 146 236 2de 4fi 5gh 79b 7ac 7fh 89c 8ab 8gi 9ad
  bcd efg ehi
012 034 056 135 146 29a 2bc 367 458 79b 7ac 89c 8de afi bgh dfh
  dgi efg ehi
```

Although there is no symmetric configuration 20_3 without a blocking set, increases in computer power allowed us to extend the enumeration of symmetric configurations to the case where $v = 20$ and this information is summarised in Table 2. Our enumeration, in common with our previous results [12], was carried out using the program confibaum as used in [6].

We are grateful to G. Brinkmann for this program and for assistance in our previous enumeration.

The enumeration confirms the fact that there is no symmetric configuration 20_3 without a blocking set. For completeness we describe here the properties enumerated in Table 2, following the notation of [6]. For a configuration $\mathcal{X}$, an *automorphism* is a permutation of the points and blocks of $\mathcal{X}$ which preserves incidence. The *dual* of $\mathcal{X}$ is the configuration obtained by reversing the roles of the points and blocks of $\mathcal{X}$. If $\mathcal{X}$ is isomorphic to its dual, we say it is *self-dual*, and an isomorphism between $\mathcal{X}$ and its dual is an *anti-automorphism*. An anti-automorphism of $\mathcal{X}$ of order two is called a *polarity*, and a configuration admitting such an isomorphism is *self-polar*. The group of all automorphisms of $\mathcal{X}$ (preserving the roles of points and blocks) is denoted by $\mathrm{Aut}(\mathcal{X})$, and the group of all automorphisms and anti-automorphisms by $A(\mathcal{X})$. If $\mathrm{Aut}(\mathcal{X})$ acts transitively on the points of $\mathcal{X}$ then we say $\mathcal{X}$ is *point-transitive*. A *flag* of $\mathcal{X}$ is an ordered pair (p, B) with $p \in B$; if $\mathrm{Aut}(\mathcal{X})$ acts transitively on the set of flags then we say $\mathcal{X}$ is *flag-transitive*; if $A(\mathcal{X})$ acts transitively on the set of flags regarded as *unordered* pairs, then we say $\mathcal{X}$ is *weakly flag-transitive*. A *cyclic* configuration $\mathcal{X}$ is one admitting a cyclic subgroup of $\mathrm{Aut}(\mathcal{X})$ acting regularly on points.

Note that for consistency with previously published results, the counts in Table 2 include disconnected configurations.

The next case to consider is $v = 21$. A 21_3 configuration without a blocking set can be constructed from three 7_3 configurations by Theorem 2.11. Because the automorphism group of the Fano plane is flag-transitive, all systems constructed by this method are isomorphic. We show that this is the unique system of this order without a blocking set. From Theorem 2.13, any such system is 2-connected.

The first observation to make is that a cubic bipartite graph with edge connectivity 2 and edge cutset $\{ab, cd\}$ must take the form illustrated in Figure 4. In the diagram, the circles represent the components C_1, C_2 following the edge cut and the black/white colouring of the vertices represents the bipartition of the graph.

Suppose now that the graph in Figure 4 is the Levi graph of a symmetric configuration 21_3. Say the components C_1, C_2 following the edge cut have respective orders n_1 and n_2, with $n_1 + n_2 = 42$. Then C_1 has $n_1 - 2$ vertices of valency 3, and two vertices (a and d) of valency 2. In other words, it is a subcubic bipartite graph with $3n_1/2 - 1$ edges. A similar argument holds for C_2, where the distinguished vertices of valency 2 are b and c.

The problem of constructing all cubic bipartite graphs with edge connectivity 2 can therefore be reduced to finding all possible components C_1, C_2. Note that a component is not necessarily an edge-deleted Levi graph of some configuration; this will be the case for C_1, for example, if and only if the distance between the distinguished vertices a and d is at least 5. But these vertices may be at distance 3 or even 1. However the component can contain no 4-cycles. By using the `genbg` utility provided in the `nauty` package [21], we may use a computer to construct all possible components. This computer search shows that the smallest possible one of these has order 14 and is unique; it is an edge-deleted Heawood graph. At order 16 there are three possible components: one with a, d at distance 5 which is an edge-deleted Levi graph of the 8_3 configuration; one with a, d at distance 3 and one with a, d adjacent.

In principle then, all cubic bipartite graphs with edge connectivity 2, girth at least 6 and order 42 can be constructed by finding all possible components C_1, C_2 such that $n_1 + n_2 = 42$ and joining their distinguished vertices as in Figure 4. The join can be done in two (possibly) nonisomorphic ways and is subject to the constraint that at least one of C_1, C_2 must have its distinguished vertices nonadjacent (to avoid creating a 4-cycle).

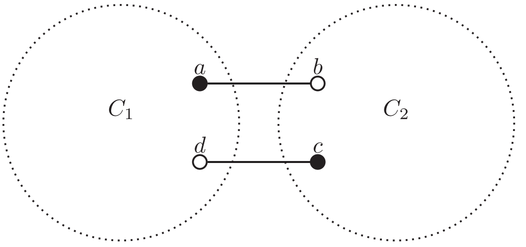

FIGURE 4 A cubic bipartite graph with edge connectivity 2

We therefore proceed as follows. For $n = 14, 16, \ldots, 28$ we generate using `genbg` all subcubic bipartite graphs of order n and girth at least 6 with $3n/2 - 1$ edges. Then the idea is that we connect up a graph of order n with a graph of order $42 - n$ as above, subject to the constraint noted. The resulting cubic graph will have girth at least 6; this process therefore generates the entire population of 2-edge-connected cubic bipartite graphs of order 42. Any blocking set free configuration at $v = 21$ must have a Levi graph within this population.

Although there are a large number of possible components, it turns out that with modern computers the generation of the components and hence the enumeration of all possible Levi graphs of configurations 21_3 could be completed. Exactly one of the resulting Levi graphs arose from a configuration which failed to have a blocking set. It is illustrated in Figure 1 and its blocks are as follows.

```
012 034 056 135 146 236 24l 58f 79c 7ak 7bl 89a 8ck 9bk abc deh
  dfj dgi egj eil fgh hij
```

We therefore have the following result.

Theorem 2.15. *There is a unique symmetric configuration 21_3 having no blocking set; it is the configuration obtained by using three Fano planes in the construction of Theorem 2.11.*

As noted above, the symmetric configuration 22_3 with no blocking set illustrated in Figure 1 was found by Dorwart and Grünbaum [11]. In fact we can show that this also is the unique such configuration. We use the same procedure as for the 21_3 configuration, but the search can be considerably shortened by the following simple lemma.

Lemma 2.16. *Let $v \geq 8$ be an even number. If a symmetric configuration v_3 contains no blocking set, then its Levi graph is non-Hamiltonian.*

Proof. Suppose that the Levi graph contains a Hamiltonian cycle $p_0B_0p_1B_1 \cdots p_{v-1}B_{v-1}p_0$ where p_i and B_i, respectively, represent point and block vertices, $i = 0, 1, \ldots, v - 1$. Colour the even-numbered points $p_0, p_2, \ldots, p_{v-2}$ red and the odd-numbered points blue. Since v is even, no block is monochromatic and so the even-numbered points form a blocking set for the configuration. □

Lemma 2.16 and Theorem 2.13 show that if a symmetric configuration 22_3 has no blocking set, its Levi graph must be a 2-connected non-Hamiltonian cubic bipartite graph of order 44.

Thus an enumeration of blocking set free configurations on 22 points can be achieved by an exhaustive enumeration of such graphs.

We use the same basic search methodology as in the 21_3 case, but this time we extend the generation of the components C_1, C_2 up to order 30. To guarantee that the resulting graph of order 44 will be non-Hamiltonian, we require that at least one of C_1, C_2 must fail to have a Hamiltonian path between its distinguished vertices. (To check the existence of a Hamiltonian path, we create an augmented graph in which the two distinguished vertices of valency 2 are joined to a new vertex; then the augmented graph is Hamiltonian if and only if there is a Hamiltonian path between the distinguished vertices in the original graph. This technique allows us to use the well-tested `cubhamg` utility in the `nauty` package, rather than writing new software for the Hamiltonicity test.)

Restricting our search to pairs C_1, C_2 such that at least one component fails to have a Hamiltonian path between the distinguished vertices gives a very substantial reduction in the number of component pairs to be considered. We were thus able to complete the enumeration of the 2-connected non-Hamiltonian cubic bipartite graphs of order 44, and found that only one of these is the Levi graph of a blocking set free configuration 22_3. Thus we have the following result.

Theorem 2.17. *There is a unique symmetric configuration* 22_3 *having no blocking set; it is the configuration of Dorwart and Grünbaum* [11].

Next, a 25_3 configuration without a blocking set can be constructed by Theorem 2.10 using either two 13_3s or a 7_3 with one of the 19_3s. Again in [16], Gropp reports that there are at least 19 such configurations. With the assistance of computers in a similar way to the construction of the 19_3 configurations, in fact we find 23 isomorphism classes of configurations 25_3 arising from Theorem 2.10 in this way. The blocks of these are given in the appendix.

All of these systems have connectivity 3 and we now know that there is at least one further system which is 2-connected; thus 25 is the smallest order for which there exist both 3-connected and 2-connected blocking set free systems. Using Theorem 2.13 we can now make an enumeration of 3-connected symmetric configurations v_3 without a blocking set for $v \in \{7, 13, 19, 25, 31, 37, 43\}$. We do this by repeated application of the $v + v' - 1$ construction in all possible ways, and reducing the resulting configurations to a set of isomorphism class representatives. The results are shown in Table 3.

3 | STRONG COLOURINGS

In this section we turn our attention to the strong chromatic number χ_s of a symmetric configuration, and also investigate its relationship to the weak chromatic number χ_w. Our first observation is that the strong chromatic number of a configuration is equal to the chromatic number of its associated graph. Since the associated graph is regular of valency 6 and contains triangles, it follows from Brooks' Theorem that $\chi_s \in \{3, 4, 5, 6, 7\}$, and $\chi_s = 7$ if and only if the associated graph is a complete graph; that is to say, the configuration is the Fano plane.

The first case to consider is $\chi_s = 3$. An immediate observation is that each block of the configuration must contain exactly one point from each of the three colour classes, and so $v \equiv 0 \pmod 3$. By colouring two classes in the strong colouring (say) red and the third blue, we see that $\chi_s = 3$ implies $\chi_w = 2$.

TABLE 3 Numbers of 3-connected blocking set free configurations v_3

v	Configurations	Self-dual	Self-polar
7	1	1	1
13	1	1	1
19	4	2	2
25	23	5	5
31	182	14	14
37	1747	45	45
43	19,485	145	145

A nice description of strongly 3-chromatic configurations is as follows. From the associated graph of the configuration, form the subgraph induced by the points from any two of the three colour classes. It is easy to see that this induced subgraph is a cubic bipartite graph (not necessarily connected), and a given strong 3-colouring of a configuration gives rise to three cubic bipartite graphs in this way by deleting each of the colour classes. Given any cubic bipartite graph Γ, it is natural to ask whether Γ can arise in this way. Our next result answers this in the affirmative.

Theorem 3.1. *Let $m \geq 3$ and let Γ be a cubic bipartite graph of order $2m$. Then there exists a strongly 3-chromatic symmetric configuration $\mathcal{X}$ on $3m$ points, and a strong 3-colouring of $\mathcal{X}$, such that the induced subgraph of the associated graph of $\mathcal{X}$ formed by deleting the points of one colour class is isomorphic to Γ.*

Proof. Our aim is to construct a new 6-regular graph Γ' on $3m$ vertices to be the associated graph of our configuration $\mathcal{X}$. We begin by creating three sets of vertices V_1, V_2, V_3, each of order m. Between the vertices of V_1 and V_2 we add edges such that the induced subgraph on $V_1 \cup V_2$ is isomorphic to Γ. We now note that by [7], the edges of Γ can be decomposed into a collection of m copies of the graph $3K_2$, that is, a collection of m sets of three disjoint edges. Each of the m sets of three edges contains exactly six vertices; we construct Γ' by joining each of the m vertices in V_3 to all the vertices in exactly one of these sets.

Since Γ' is a 6-regular tripartite graph, any decomposition of its edge set into triangles will yield a strongly 3-chromatic configuration on $3m$ points, where the colour classes are the sets V_1, V_2, V_3. A suitable triangle decomposition is given by using each edge between vertices in V_1 and V_2 together with the two edges joining its endpoints to a vertex in V_3. By construction, the configuration $\mathcal{X}$ represented by this decomposition has the required properties, taking the colour class assigned to V_3 as the one to be deleted. □

In general, the three cubic bipartite graphs formed by deleting a colour class from a strongly 3-chromatic configuration in this way will not be isomorphic. Another natural question is whether we can construct strongly 3-chromatic configurations in such a way that, with a suitable colouring, the resulting colour class deleted graphs are actually isomorphic. It turns out that we can do this for any $v \geq 9$ which is a multiple of 3.

Theorem 3.2. *Let $s \geq 3$ and let $v = 3s$. Then there is a cubic bipartite graph Γ of order $2s$, and a strongly 3-chromatic symmetric configuration $\mathcal{X}$ on v points, such that deleting any of the three colour classes in a suitable colouring of $\mathcal{X}$ we obtain a graph isomorphic to Γ.*

Proof. We begin by defining a suitable cubic bipartite graph Γ. Let the vertex set of Γ consist of $\{a_0, a_1, ..., a_{s-1}\} \cup \{b_0, b_1, ..., b_{s-1}\}$. There is an edge from a_i to b_j if and only if $i - j \in \{-1, 0, 1\}$, where of course the arithmetic is modulo s. Now we extend Γ to a 6-regular graph Γ', and colour the edges in a particular way. To create Γ', create a new vertex set $\{c_0, c_1, ..., c_{s-1}\}$ and join edges c_i to a_j and c_i to b_j exactly as in Γ. A triangle decomposition in Γ' can be defined as follows. For each $i = 0, 1, ..., s - 1$, colour the edges in Γ' according to the following rules:

- Edges from a_i to b_{i-i}, b_i to c_i and c_i to a_{i+1} are coloured red.
- Edges from a_i to b_i, b_i to c_{i+1} and c_i to a_{i-1} are coloured green.
- Edges from a_i to b_{i+1}, b_i to c_{i-1} and c_i to a_i are coloured blue.

Then the monochromatic triangles in the above edge-coloring form a triangle decomposition of Γ'. The configuration $\mathcal{X}$ represented by this decomposition is strongly 3-chromatic (since Γ' is tripartite) and deleting any of the three sets in the tripartition leaves a graph isomorphic to Γ. □

Note that the symmetric configuration constructed in the above theorem is resolvable, the sets of monochromatic triangles of the three colours forming the resolution classes. The graph Γ' is a Cayley graph of the group $\mathbb{Z}_3 \times \mathbb{Z}_s$.

We next turn our attention to the case $\chi_s = 4$. It is easy to see that a strongly 4-chromatic configuration is weakly 2-chromatic; if we strongly colour the configuration with colours 1, 2, 3, 4 then we can colour the points in colour classes 1 and 2 blue, and the remainder red. Then no block is monochromatic.

In Table 4 we give computer calculations of the strong chromatic numbers of all connected configurations with $v \leq 15$; the numerical evidence is that the case $\chi_s = 4$ seems to be the most common. Indeed, our next result shows that we can construct a symmetric configuration with $\chi_s = 4$ for all $v \geq 8$.

Theorem 3.3. *There exists a strongly 4-chromatic configuration v_3 for all $v \geq 8$.*

Proof. The proof is similar to that of Theorem 2.3. We again use Martinetti's extension operation, though the replacement of blocks is different from that done in Theorem 2.3. First observe from Table 4 that a strongly 4-chromatic configuration v_3 exists for $8 \leq v \leq 12$.

Let $v = 3s$ where $s \geq 4$, and let $V = \{a_i, b_i, c_i: 0 \leq i \leq s - 1\}$. Let the blocks of the symmetric configuration v_3 be the sets $\{a_i, b_i, c_{i+1}\}, \{a_i, b_{i+1}, c_i\}$ and $\{a_{i+1}, b_i, c_i\}, 0 \leq i \leq s - 1$.

Now suppose that $v \equiv 1 \pmod 3$, $v \geq 13$. Construct a configuration $(v - 1)_3$ as above. Introduce a new point ∞_0 and use the extension operation, replacing the blocks $\{a_0, b_0, c_1\}$ and $\{a_1, b_1, c_2\}$ by blocks $\{\infty_0, b_0, c_2\}$, $\{\infty_0, a_0, c_1\}$ and $\{\infty_0, a_1, b_1\}$. Next suppose that $v \equiv 2 \pmod 3$, $v \geq 14$. Construct a configuration $(v - 1)_3$ as above. Introduce a new point ∞_1 and again use the extension operation, replacing the blocks $\{a_0, b_1, c_0\}$ and $\{a_1, b_2, c_1\}$ by blocks $\{\infty_1, a_0, b_2\}$, $\{\infty_1, b_1, c_0\}$ and $\{\infty_1, a_1, c_1\}$.

TABLE 4 Strong chromatic numbers of connected configurations v_3

v	Total	$\chi_s = 3$	$\chi_s = 4$	$\chi_s = 5$	$\chi_s = 6$	$\chi_s = 7$
7	1	0	0	0	0	1
8	1	0	1	0	0	0
9	3	1	1	1	0	0
10	10	0	3	7	0	0
11	31	0	21	9	1	0
12	229	4	161	64	0	0
13	2036	0	1451	584	1	0
14	21,398	0	17,342	4053	3	0
15	245,341	251	234,139	10,938	13	0

Finally, suppose that $v \equiv 0 \pmod 3$, $v \geq 15$. Construct a configuration $(v-1)_3$ as above. Introduce a new point ∞_2 and again use the extension operation, replacing the blocks $\{a_1, b_0, c_0\}$ and $\{a_2, b_1, c_1\}$ by blocks $\{\infty_2, a_2, c_0\}$, $\{\infty_2, a_1, b_0\}$ and $\{\infty_2, b_1, c_1\}$.

In all cases it is clear that the symmetric configurations so constructed have a strong colouring with four colours and therefore in the first two cases are strongly 4-chromatic. It remains to prove that in the case where $v \equiv 0 \pmod 3$ it is not 3-chromatic. Suppose that it is and that in the block $\{\infty_0, b_0, c_2\}$, ∞_0 receives colour 1, b_0 receives colour 2 and c_0 receives colour 3. Then in the block $\{\infty_0, a_0, c_1\}$, a_0 and c_1 receive colours 2 and 3 in some order and likewise in the block $\{\infty_0, a_1, b_1\}$, a_1 and b_1 receive colours 2 and 3 in some order, giving four possibilities in all. However in all cases either a_1 and c_1 or b_1 and c_1 receive the same colour, giving a contradiction. □

Before considering the next case $\chi_s = 5$, we state and prove the following theorem which gives the strong chromatic number of certain cyclic configurations.

Theorem 3.4. *Let $v \geq 7$ and let C_v be the cyclic configuration on v points generated by the block $\{0, 1, 3\}$ under the mapping $i \mapsto i + 1 \pmod v$. Then*

$$\chi_s(C_v) = \begin{cases} 7 & \text{if } v = 7, \\ 6 & \text{if } v = 11, \\ 5 & \text{if } v \equiv 1, 2, 3 \pmod 4, \quad v \notin \{7, 11\}, \\ 4 & \text{if } v \equiv 0 \pmod 4. \end{cases}$$

Proof of this theorem is facilitated by the following lemma.

Lemma 3.5. *Let $v \geq 7$ and let C_v be the cyclic configuration on v points generated by the block $\{0, 1, 3\}$. If v can be expressed in the form $4a + 5b$ where a, b are nonnegative integers, then $\chi_s(C_v) \leq 5$.*

Proof. We consider the points of C_v to be elements of the cyclic group $\mathbb{Z}_v$. The colours of the points will be taken from the set $\{0, 1, 2, 3, 4\}$. Each point i is assigned colour $c(i)$ as follows.

$$c(i) = \begin{cases} i \bmod 4 & \text{if } i < 4a, \\ (i - 4a) \bmod 5 & \text{if } i \geq 4a. \end{cases}$$

So listing the elements of $\mathbb{Z}_v$ from 0 to $v - 1$ in order, the assignment of colours looks like

$$0123 \quad 0123 \ldots 0123 \quad 01234 \quad 01234 \ldots 01234.$$

It is easy to see that if $b > 0$ this represents a strong 5-colouring of C_v; and if v is divisible by 4 we can write $v = 4a$ and it is a 4-colouring. □

Proof of Theorem 3.4. The block $\{0, 1, 3\}$ shows that 0, 1 and 3 must be assigned different colours; then the blocks $\{1, 2, 4\}$, $\{2, 3, 5\}$ and $\{v - 1, 0, 2\}$ show that 2 must be assigned a fourth colour. So for any $v \geq 7$, $\chi_s(C_v) \geq 4$.

If v can be written in the form $4a + 5b$, then Lemma 3.5 applies and so $\chi_s(C_v)$ will equal 4 if $v \equiv 0 \pmod 4$. If $v \not\equiv 0 \pmod 4$, then by the paragraph above and the proof of Lemma 3.5, the assignment of colours in a strong 4-colouring would have to be 0123 0123 ... 0123 which is impossible because the points cannot be split into groups of 4. So $\chi_s(C_v) = 5$.

The only values of $v \geq 7$ which cannot be written in the form $4a + 5b$ are 7 and 11. If $v = 7$ then the associated graph of C_v is the complete graph K_7 and this has chromatic number 7. If $v = 11$ then Lemma 3.5 cannot be applied, and computer testing shows that $\chi_s(C_{11}) = 6$. In fact as Table 4 shows, this is the unique 6-chromatic configuration 11_3. □

The case $\chi_s = 5$ is interesting. Theorem 3.4 shows that symmetric configurations v_3 with $\chi_s = 5$ exist for all $v \equiv 1, 2, 3 \pmod 4$, $v \notin \{7, 11\}$ and Table 4 shows that such a configuration also exists for $v = 11$ but not $v = 7$. It remains to determine existence for $v \equiv 0 \pmod 4$, which is more appropriate for us to do later in Theorem 3.8.

All examples of strongly 5-chromatic configurations v_3 with $v \leq 15$ have $\chi_w = 2$, and indeed all other examples we have seen have $\chi_w = 2$ (in other words, the configuration contains a blocking set). However, we have been unable to find a proof of this, and so the existence of a symmetric configuration with $\chi_s = 5$ and $\chi_w = 3$ remains an open question. As a partial result in this direction, we can show that all configurations which are "almost" strongly 4-colourable have weak chromatic number 2.

Theorem 3.6. *Suppose that we have a strongly 5-chromatic configuration v_3 in which all but at most two points can be coloured using four colours. Then the weak chromatic number of the configuration is 2.*

Proof. First, note that in any 5-colouring each of the v blocks is coloured with one of the $\binom{5}{3} = 10$ possible sets of three colours; and each of these sets must appear at least once if the weak chromatic number is 3. (If a set of three colours does not appear in any block, we can assign blue to these three and red to the other two to get a weak 2-colouring.)

Now suppose that we can assign four colours (say 1, 2, 3 and 4) to $v - 1$ points so that no colour is repeated in a block. Clearly we can assign a fifth colour 5 to the remaining point, and in this 5-colouring at least three sets of three colours must fail to appear in any block, since there are six possible sets containing this colour but only three blocks containing the single vertex with this colour. Thus the configuration has weak chromatic number 2.

If we can only assign four colours to $v - 2$ points the position is more awkward. Let the two uncoloured points be a and b. Suppose first that a and b do not appear in the same block. We seek an assignment of two of the existing colours 1, 2, 3, 4 to red and the remaining two to blue, such that we can choose red or blue for a and b to obtain a weak 2-colouring. There are exactly three ways to do this initial red/blue assignment: 12/34, 13/24 and 14/23. We shall call an assignment *compatible* with a if it leaves a possible red/blue choice for a such that no monochromatic block is created. Since a appears in three blocks, it is easy to see that at most one of the three possible assignments is not compatible with a. For example, if the colours of the other points in the blocks containing a are $\{1, 2\}$, $\{3, 4\}$ and $\{1, 4\}$, then the assignment 12/34 is incompatible with a but the assignments 13/24 and 14/23 are compatible. Since the same argument holds for b, at least one possible assignment is compatible with a and b and so the configuration has a weak 2-colouring.

If a and b do appear in the same block, then each has two other blocks in which it appears. In this case, not only is there an assignment compatible with both a and b, but also the choice of red/blue for a and b may be made freely. So we can choose red for a and blue for b and again there is a weak 2-colouring. □

Now we come to the case $\chi_s = 6$. Table 4 shows that this is uncommon; of the 269,049 connected configurations v_3 with $8 \leq v \leq 15$, only 18 are strongly 6-chromatic. These are given in the appendix. Nevertheless, we are able to deduce the existence of strongly 6-chromatic configurations for almost all values of v as the next result shows.

Theorem 3.7. *There exists a strongly 6-chromatic connected configuration v_3 for $v = 11$ and for all $v \geq 13$.*

Proof. The cases $v = 11$ and 13 follow from Table 4. So let $v \geq 14$ and let C_7 be the cyclic configuration on seven points generated by the block $\{0, 1, 3\}$ under the mapping $i \mapsto i + 1 \pmod 7$; this is of course the unique 7_3 configuration and is strongly 7-chromatic. Now choose any connected configuration $(v - 7)_3$ and number the points from 7 to $v - 1$. By relabelling if necessary, we may assume without loss of generality that this configuration contains the block $\{7, 8, 9\}$. Now create a new configuration $\mathcal{X}$ with the blocks of these two configurations, but replacing the blocks $\{0, 1, 3\}$ and $\{7, 8, 9\}$ with $\{1, 3, 7\}$ and $\{0, 8, 9\}$. Suppose $\mathcal{X}$ can be strongly coloured with five colours. Then the colours assigned to points 1–6 together with a sixth colour for point 0 would give a strong 6-colouring for the original configuration C_7, which is impossible. Thus $\chi_s(\mathcal{X}) \geq 6$ and since $\chi_s(\mathcal{X}) \leq 6$ by Brooks' Theorem, $\mathcal{X}$ is a strongly 6-chromatic connected configuration on v points as required. □

As noted above, all the blocking set free configurations of which we are aware have $\chi_s = 6$. However, only the single example at $v = 13$ in Table 4 has $\chi_s = 6$ and $\chi_w = 3$, so there are many examples with $\chi_s = 6$ and $\chi_w = 2$.

Finally in this section we complete the proof of the existence spectrum for $\chi_s = 5$ which we earlier deferred until later. It follows the proof of Theorem 3.7 but is more intricate.

Theorem 3.8. *There exists a strongly 5-chromatic connected configuration v_3 for all $v \equiv 0 \pmod 4$, $v \geq 12$.*

Proof. Examples for $v = 12$ and 16 are the following.

```
012 034 056 135 146 237 289 48a 59b 6ab 78b 79a
012 034 056 135 146 236 278 479 57a 89b 8cd 9ef ace adf
  bcf bde
```

Now let $v \geq 20$ and let C_{11} be the cyclic configuration on 11 points generated by the block $\{0, 1, 3\}$ under the mapping $i \mapsto i + 1 \pmod{11}$; from Theorem 3.4 this is strongly 6-chromatic. Now choose any connected configuration $(v - 11)_3$ with $\chi_s = 5$ and number the points from 11 to $v - 1$. From Theorem 3.4 this is possible. By relabelling if necessary, we may assume without loss of generality that this configuration contains the block $\{11, 12, 13\}$ and that in the strong 5-colouring, these points receive colours red, yellow and blue, respectively. Now create a new configuration $\mathcal{X}$ with the blocks of these two configurations but replacing the blocks $\{0, 1, 3\}$ and $\{11, 12, 13\}$ with $\{1, 3, 11\}$ and $\{0, 12, 13\}$. Suppose $\mathcal{X}$ can be strongly coloured with four colours. Then the colours assigned to points 1–10 together with a fifth colour assigned to point 0 would give a strong 5-colouring of the original configuration C_{11}, which is impossible. Thus $\chi_s(\mathcal{X}) \geq 5$, and since $\chi_s(\mathcal{X}) \leq 6$ by Brooks' Theorem, $\mathcal{X}$ is either strongly 5- or 6-chromatic. It remains to show that it is the former by exhibiting a colouring.

Colour the blocks of the $(v - 11)_3$ configuration without the block $\{11, 12, 13\}$ with five colours red, yellow, blue, green and white, respecting that colours have already been assigned to points 11, 12 and 13. Colour the remaining points as follows: 4 and 8 red; 2 and 9 yellow; 3 and 7 blue; 0, 1 and 5 green; 6 and 10 white. □

4 | OPEN QUESTIONS

We gather here some of the interesting open questions arising from this study. The first of these relates to symmetric configurations 25_3 without a blocking set. We now have enumerations of all symmetric configurations v_3 for $7 \leq v \leq 20$ and all 3-connected symmetric configurations without a blocking set for $7 \leq v \leq 43$. There are unique configurations 21_3 and 22_3 without a blocking set, both necessarily 2-connected, and a 2-connected configuration 25_3 without a blocking set is known. The question remains whether this is unique.

The second problem is to extend the work on the sizes of minimal blocking sets, possibly along the lines of Theorems 2.4 and 2.5. In particular it would be interesting to find constructions of symmetric configurations v_3 whose minimal blocking set has maximum cardinality, that is, $(v - 1)/2$ if v is odd and $v/2$ if v is even. The admittedly limited evidence from Table 1 suggests that such configurations exist except for $v = 7$ (where there is no blocking set) and $v = 14$, though amongst the set of all symmetric configurations they may be relatively rare. However, given the long history of blocking set free symmetric configurations, finding those with only minimal blocking sets of maximum cardinality may also be quite challenging.

The third problem concerns the relationship between the strong and the weak chromatic numbers. We have observed that if $\chi_s = 3$ or 4 then $\chi_w = 2$ and that there are configurations with (χ_s, χ_w) equal to both $(6, 2)$ and $(6, 3)$. However all of the known systems with $\chi_s = 5$ have $\chi_w = 2$. So we ask does there exist a symmetric configuration v_3 with strong chromatic number 5 and weak chromatic number 3? Equivalently, does every blocking set free configuration have strong chromatic number 6?

Finally, as we observed, a given strong 3-colouring of a strongly 3-chromatic configuration gives rise to three cubic bipartite graphs by deleting each of the colour classes from the associated graph of the configuration. Denote these graphs by Γ_1, Γ_2 and Γ_3. In Theorem 3.1 we proved that one of these graphs, say Γ_1, can be any cubic bipartite graph. Now suppose that Γ_1, Γ_2 and Γ_3 are all specified. Does there exist a symmetric configuration v_3 whose three cubic bipartite graphs constructed as above are isomorphic to Γ_1, Γ_2 and Γ_3? If not, what are the constraints on these three graphs for this to be possible? The case where Γ_1, Γ_2 and Γ_3 are isomorphic would be of particular interest.

There are of course other problems on symmetric configurations and we hope that this paper will encourage colleagues to work on these.

ACKNOWLEDGEMENTS

The third author acknowledges support from the APVV Research Grants 15-0220 and 17-0428, and the VEGA Research Grants 1/0142/17 and 1/0238/19. We thank the anonymous referees for pointing out to us the work of Funk et al. [14] which completed the existence spectrum of weakly 3-chromatic configurations.

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How to cite this article: G. Erskine, T. Griggs, and J. Širáň, *Colouring problems for symmetric configurations with block size 3*, *J. Combin. Des*. (2021), 29, 397–423. https://doi.org/10.1002/jcd.21773

APPENDIX A

The 27 configurations with $v \leq 14$ with a minimal blocking set of size $\left\lfloor \frac{v}{2} \right\rfloor$:

```
012 034 056 135 147 246 257 367
012 034 056 135 147 248 267 368 578
012 034 056 135 178 247 268 379 469 589
012 034 056 137 158 247 268 359 469 789
012 034 056 135 146 278 29a 379 47a 589 68a
012 034 056 135 147 248 269 37a 59a 68a 789
012 034 056 135 147 248 279 36a 59a 689 78a
012 034 056 135 147 248 29a 379 58a 67a 689
012 034 056 135 147 268 279 389 49a 58a 67a
012 034 056 135 178 246 279 37a 49a 58a 689
012 034 056 135 148 257 26b 389 49a 67a 79b 8ab
012 034 056 135 179 246 278 39a 48a 59b 68b 7ab
012 034 056 135 148 257 26c 389 49a 67b 7ac 8ab 9bc
012 034 056 135 149 25c 2ab 37b 478 689 6ac 79a 8bc
012 034 056 135 167 247 2bc 389 49b 58c 68a 7ab 9ac
012 034 056 135 178 239 247 49a 58c 68b 6ac 7ab 9bc
012 034 056 135 178 247 289 37b 4ab 59c 69a 6bc 8ac
012 034 056 135 178 247 289 3bc 45c 69b 6ac 79a 8ab
012 034 056 135 179 247 269 3ac 48a 58c 6bc 78b 9ab
012 034 056 135 179 249 26c 38a 4ac 578 68b 7ab 9bc
012 034 056 135 179 268 29c 3ab 47a 49b 578 6bc 8ac
012 034 056 135 179 26a 289 3ab 478 49b 57c 6bc 8ac
012 034 056 135 179 26b 29c 3ab 47a 4bc 578 68c 89a
012 034 056 135 17c 24b 26a 38c 469 578 79b 8ab 9ac
012 034 056 137 14c 25b 26a 359 468 78c 79a 8ab 9bc
012 034 056 137 158 247 2ab 368 49a 59c 6bc 79b 8ac
012 034 056 137 189 26b 29c 3ab 45c 49a 578 68a 7bc
```

The 23 blocking set free configurations 25_3 arising from Theorem 2.10:

```
012 034 056 135 1fg 236 29c 478 4hi 5ab 6de 79b 7ac 89a 8bc dfh dgi efi
ejk glo hmn jln jmo klm kno
012 034 056 135 146 29a 2bc 367 4mn 5lo 79b 7ac 89c 8de 8jk afi bgh dfh
dgi efg ehi jln jmo klm kno
012 034 056 135 146 236 2jk 4lo 5mn 79b 7ac 7ln 89c 8de 8mo 9aj afi bcj
bgh dfh dgi efg ehi klm kno
012 034 056 135 146 236 2de 49c 5ab 79b 7ac 7fg 89a 8bc 8hi dfh dgi efi
ejk glo hmn jln jmo klm kno
012 034 056 135 146 236 2de 49c 5ab 79b 7ac 7fi 89a 8bc 8jk dfg dhi efh
egi glo hmn jln jmo klm kno
012 034 056 135 146 236 28k 4lo 5mn 78j 79b 7ac 89c 9aj afi bcj bgh dfh
dgi dln efg ehi emo klm kno
012 034 056 135 146 236 278 49c 5ab 7ac 7de 89a 8bc 9fg bhi dfh dgi efi
ejk glo hmn jln jmo klm kno
012 034 056 135 146 236 278 49c 5ab 7ac 7de 89a 8bc 9fi bjk dfg dhi efh
egi glo hmn jln jmo klm kno
012 034 056 135 146 236 278 49c 5ab 7ac 7fk 89a 8bc 9ln bmo def dgh dij
egi ehj fgj hlo imn klm kno
012 034 056 135 1de 236 245 4fg 6hi 79b 7ac 7fh 89c 8ab 8gi 9ad bcd efi
ejk glo hmn jln jmo klm kno
012 034 056 135 1de 236 245 4fg 6hi 79b 7ac 7fi 89c 8ab 8jk 9ad bcd efh
egi glo hmn jln jmo klm kno
012 034 056 135 1ab 236 245 4jk 69c 79a 7bc 7de 89b 8ac 8fi dfg dhi efh
egi glo hmn jln jmo klm kno
012 034 056 135 1ef 236 245 4gi 6jk 79b 7ac 7fi 89c 8ab 8lo 9ad bcd deg
ehi fgh hmn jln jmo klm kno
012 034 056 135 1ef 236 245 4gj 6kl 79b 7ac 7kn 89c 8ab 8lm 9ad bcd dho
egh eij fgi fhj imn kmo lno
012 034 056 135 146 236 2dk 4lo 5mn 79b 7ac 7lm 89c 8ab 8no 9ak bck dgh
dij egi ehj eln fgj fhi fmo
012 034 056 135 146 236 27o 4mn 5dl 79b 7ac 89c 8ab 8lm 9ak bck dgh dij
egi ehj ekn fgj fhi fmo lno
012 034 056 135 1bc 236 245 4jk 69a 79b 7ac 7lm 89c 8de 8no afi bgh dfh
dgi efg ehi jln jmo klo kmn
012 034 056 135 1de 236 245 4no 69c 79a 7bc 7jk 89b 8ac 8lm afi bgh dfh
dgi efg ehi jln jmo klo kmn
012 034 056 135 18b 236 245 49c 67a 7bc 7jk 8ac 8lm 9de 9no afi bgh dfh
dgi efg ehi jln jmo klo kmn
012 034 056 135 1fh 236 245 4gi 69e 7ab 7cd 7jk 8ac 8bd 8lm 9ad 9no bfi
cgh efg ehi jln jmo klo kmn
012 034 056 135 146 2bc 2de 369 45a 79b 7ac 7fi 89c 8ab 8jk dfg dhi efh
egi glo hmn jln jmo klm kno
012 034 056 135 146 2bc 2jk 367 458 79b 7ac 89c 8de 9lm afi ano bgh dfh
dgi efg ehi jln jmo klo kmn
012 034 056 135 146 29a 2bc 367 458 79b 7ac 8de 8jk 9lm afi bgh cno dfh
dgi efg ehi jln jmo klo kmn
```

The 18 strongly 6-chromatic configurations with $v \leq 15$:

```
012 034 056 135 147 248 279 36a 59a 689 78a
012 034 056 135 146 236 278 49c 5ab 79b 7ac 89a 8bc
012 034 056 135 146 236 247 589 7ad 7bc 8ac 8bd 9ab 9cd
012 034 056 135 146 28d 29c 368 457 79a 7bc 8ab 9bd acd
012 034 056 135 146 29b 2cd 368 457 79a 7bc 89d 8ac abd
012 034 056 135 146 28d 29c 36e 457 789 7bc 8ab 9ae acd bde
012 034 056 135 147 239 245 6ae 6cd 789 7bc 8ab 8ce 9ad bde
012 034 056 135 146 24c 25e 38a 6ce 78d 79e 7ab 89b 9ad bcd
012 034 056 135 146 25c 26e 38a 4ce 78d 79e 7ab 89b 9ad bcd
012 034 056 135 146 29b 2ae 368 45c 789 7ab 7de 8bd 9ce acd
012 034 056 135 146 28a 2de 36d 45e 78b 79e 7ac 89c 9ab bcd
012 034 056 135 146 27c 28a 36d 45e 78b 79e 89c 9ab ade bcd
012 034 056 135 146 236 24e 5de 78b 79e 7ac 89c 8ad 9ab bcd
012 034 056 135 146 24d 25e 36e 78b 79e 7ac 89c 8ad 9ab bcd
012 034 056 135 146 236 28c 45e 78b 79e 7ac 89d 9ab ade bcd
012 034 056 135 146 25c 2be 368 48c 789 7ab 7de 9ae 9bd acd
012 034 056 135 147 23b 245 68d 6ae 79e 7ac 89a 8bc 9bd cde
012 034 056 135 146 27d 2be 369 45c 789 7ab 8bc 8de 9ae acd
```